

# Distribution of crossings, nestings and alignments of two edges in matchings and partitions

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**Abstract.** We construct an involution on set partitions which keeps track of the numbers of crossings, nestings and alignments of two edges. We derive then the symmetric distribution of the numbers of crossings and nestings in partitions, which generalizes Klazar's recent result in perfect matchings. By factorizing our involution through bijections between set partitions and some path diagrams we obtain the continued fraction expansions of the corresponding ordinary generating functions.

## 1 Introduction

A *partition* of  $[n] := \{1, 2, \dots, n\}$  is a collection of disjoint nonempty subsets of  $[n]$ , called *blocks*, whose union is  $[n]$ . A (perfect) *matching* of  $[2n]$  is a partition of  $[2n]$  in  $n$  two-element blocks. Denote by  $\Pi_n$  the set of the partitions of  $[n]$  and by  $\mathcal{M}_{2n}$  the set of the matchings of  $[2n]$ . A partition  $\pi$  with  $k$  blocks is written  $\pi = B_1 - B_2 - \dots - B_k$ , where the blocks are ordered in the increasing order of their minimum elements and, within each block, the elements are written in the numerical order.

It is convenient to identify a partition of  $[n]$  with a *partition graph* on the vertex set  $[n]$  such that there is an edge joining  $i$  and  $j$  if and only if  $i$  and  $j$  are *consecutive elements* in a same block. We note such an edge  $e$  as a pair  $(i, j)$  with  $i < j$ , and say that  $i$  is the *left-hand endpoint* of  $e$  and  $j$  is the *right-hand endpoint* of  $e$ . A *singleton* is the element of a block which has only one element, so a *singleton* corresponds to an isolated vertex in the graph. Conversely, a graph on the vertex set  $[n]$  is a partition graph if and only if each vertex is the left-hand (resp. right-hand) endpoint of at most one edge. By convention, the vertices  $1, 2, \dots, n$  are arranged on a line in the increasing order from left to right and an edge  $(i, j)$  is drawn as an arc above the line. An illustration is given in Figure 1.



Figure 1: Graph of the partition  $\pi = \{1, 9, 10\} - \{2, 3, 7\} - \{4\} - \{5, 6, 11\} - \{8\}$

Given a partition  $\pi$  of  $[n]$ , two edges  $e_1 = (i_1, j_1)$  and  $e_2 = (i_2, j_2)$  of  $\pi$  is said to form:

- (i) a *crossing* with  $e_1$  as the *initial edge* if  $i_1 < i_2 < j_1 < j_2$  ;
- (ii) a *nesting* with  $e_2$  as *interior edge* if  $i_1 < i_2 < j_2 < j_1$ ;
- (iii) an *alignment* with  $e_1$  as *initial edge* if  $i_1 < j_1 \leq i_2 < j_2$ .

An illustration of these notions is given in Figure 2.

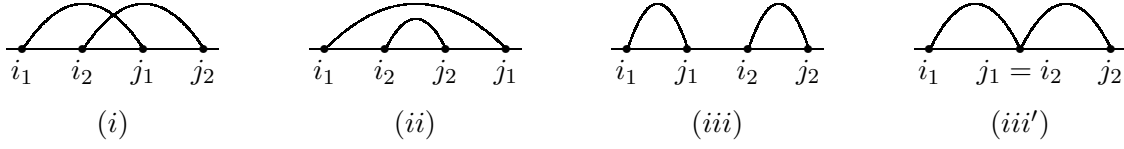


Figure 2: Crossing, nesting and alignments of two edges

We denote by  $\text{cr}(\pi)$ ,  $\text{ne}(\pi)$  and  $\text{al}(\pi)$  the numbers of crossings, nestings and alignments of two edges in  $\pi$ , respectively. Furthermore, consider a block  $B$  of  $\pi$  whose cardinal is  $\geq 2$ . An element of  $B$  is:

- (i) an *opener* if it is the least element of  $B$ ,
- (ii) a *closer* if it is the greatest element of  $B$ ,
- (iii) a *transient* if it is neither the least nor greatest elements of  $B$ .

In the graph of  $\pi$ , the edges around an opener, closer, singleton or transient are illustrated in Figure 3.



Figure 3: opener, closer, singleton and transient in a partition graph

The sets of openers, closers, singletons and transients of  $\pi$  will be denoted by  $\mathcal{O}(\pi)$ ,  $\mathcal{C}(\pi)$ ,  $\mathcal{S}(\pi)$  and  $\mathcal{T}(\pi)$ , respectively. The 4-tuple  $\lambda(\pi) = (\mathcal{O}(\pi), \mathcal{C}(\pi), \mathcal{S}(\pi), \mathcal{T}(\pi))$  is called the *type* of  $\pi$ .

For the partition  $\pi$  in Figure 1, we have  $\text{cr}(\pi) = 2$ ,  $\text{ne}(\pi) = 5$  and  $\text{al}(\pi) = 8$ . Moreover,  $\mathcal{O}(\pi) = \{1, 2, 5\}$ ,  $\mathcal{C}(\pi) = \{7, 10, 11\}$ ,  $\mathcal{S}(\pi) = \{4, 8\}$  and  $\mathcal{T}(\pi) = \{3, 6, 9\}$ .

**Definition 1.1** A 4-tuple  $\lambda = (\mathcal{O}, \mathcal{C}, \mathcal{S}, \mathcal{T})$  of subsets of  $[n]$  is a partition type of  $[n]$  if there exists a partition of  $[n]$  whose type is  $\lambda$ . Denote by  $\Pi_n(\lambda)$  the set of partitions of type  $\lambda$ , i.e.,

$$\Pi_n(\lambda) = \{\pi \in \Pi_n : \lambda(\pi) = \lambda\}.$$

In particular, a partition type  $\lambda$  is a matching type if  $\lambda = (\mathcal{O}, \mathcal{C}) := (\mathcal{O}, \mathcal{C}, \emptyset, \emptyset)$ . Denote by  $\mathcal{M}_{2n}(\gamma)$  the set of matchings of type  $\gamma$ , i.e.,

$$\mathcal{M}_{2n}(\gamma) = \{\alpha \in \mathcal{M}_{2n} : \mathcal{O}(\alpha) = \mathcal{O} \text{ and } \mathcal{C}(\alpha) = \mathcal{C}\}.$$

Klazar [8] has recently proved the symmetric distribution of the numbers of crossings and nestings of two edges in perfect matchings. The aim of this paper is to show that a much stronger result exists in the partitions which reduces to that of Klazar in the case of matchings. Note that Chen et al [2] have found other interesting results on the crossings and nestings in matching and partitions, while Corteel [3] has given an analogous result for permutations. Moreover we refer the reader to Krattenthaler's recent paper [9] for a more general context of related problems.

Our main result is the construction of an explicit involution on the set of partitions  $\Pi_n$ .

**Theorem 1.2** For each partition type  $\lambda$  of  $[n]$  there is an involution  $\varphi : \Pi_n(\lambda) \rightarrow \Pi_n(\lambda)$  preserving the number of alignments, and exchanging the numbers of crossings and nestings. In other words, for each  $\pi \in \Pi_n$ , we have  $\lambda(\pi) = \lambda(\varphi(\pi))$  and

$$\text{al}(\varphi(\pi)) = \text{al}(\pi), \text{cr}(\varphi(\pi)) = \text{ne}(\pi), \text{ne}(\varphi(\pi)) = \text{cr}(\pi). \quad (1.1)$$

**Corollary 1.3** For each partition type  $\lambda$  of  $[n]$ , we have

$$\sum_{\pi \in \Pi_n(\lambda)} p^{\text{cr}(\pi)} q^{\text{ne}(\pi)} t^{\text{al}(\pi)} = \sum_{\pi \in \Pi_n(\lambda)} p^{\text{ne}(\pi)} q^{\text{cr}(\pi)} t^{\text{al}(\pi)}, \quad (1.2)$$

and for each matching type  $\gamma$  of  $[2n]$ ,

$$\sum_{\alpha \in \mathcal{M}_{2n}(\gamma)} p^{\text{cr}(\alpha)} q^{\text{ne}(\alpha)} t^{\text{al}(\alpha)} = \sum_{\alpha \in \mathcal{M}_{2n}(\gamma)} p^{\text{ne}(\alpha)} q^{\text{cr}(\alpha)} t^{\text{al}(\alpha)}. \quad (1.3)$$

Summing over all partition types  $\lambda$  or matching types  $\gamma$  we get

**Corollary 1.4**

$$\sum_{\pi \in \Pi_n} p^{\text{cr}(\pi)} q^{\text{ne}(\pi)} t^{\text{al}(\pi)} = \sum_{\pi \in \Pi_n} p^{\text{ne}(\pi)} q^{\text{cr}(\pi)} t^{\text{al}(\pi)}, \quad (1.4)$$

and

$$\sum_{\alpha \in \mathcal{M}_{2n}} p^{\text{cr}(\alpha)} q^{\text{ne}(\alpha)} t^{\text{al}(\alpha)} = \sum_{\alpha \in \mathcal{M}_{2n}} p^{\text{ne}(\alpha)} q^{\text{cr}(\alpha)} t^{\text{al}(\alpha)}. \quad (1.5)$$

In particular, by taking  $t = 1$  in the above corollary, we obtain

**Corollary 1.5**

$$\sum_{\pi \in \Pi_n} p^{cr(\pi)} q^{ne(\pi)} = \sum_{P \in \Pi_n} p^{ne(P)} q^{cr(P)}, \quad (1.6)$$

and

$$\sum_{\alpha \in \mathcal{M}_{2n}} p^{cr(\alpha)} q^{ne(\alpha)} = \sum_{\alpha \in \mathcal{M}_{2n}} p^{ne(\alpha)} q^{cr(\alpha)}. \quad (1.7)$$

Identity (1.7) is due to Klazar [8]. The  $p = 1$  case of (1.7) had been previously proved by M. de Sainte-Catherine [5] and by De Médicis and Viennot [4].

Our approach can be considered as an application of the combinatorial theory of orthogonal polynomials developed by Viennot [14] and Flajolet [6]. In fact, our involution  $\varphi$  is a generalization of that used by De Médicis and Viennot [4] for matchings. A variant of this bijection on partitions has been used by Ksavrelof and Zeng [10] to prove other equinumerous results on partitions.

The paper is organized as follows: we present the involution  $\varphi$  and the proof of theorem 1.1 in section 2; in section 3 we factorize our involution through two bijections between partitions and Charlier diagrams, which permit us to derive continued fraction expansions of the ordinary generating functions with respect to the numbers of crossings and nestings of two edges in matchings and partitions.

## 2 Proof of Theorem 1.2

Let  $\pi = B_1 - B_2 - \dots - B_k$  be a partition of  $[n]$  and  $i$  an integer in  $[n]$ . The restriction  $B_j(\leq i) := B_j \cap [i]$  of the block  $B_j$  is said to be *opened* (resp. *closed* and *empty*) if  $B \not\subseteq [i]$  (resp.  $B \subseteq [i]$  and  $B \cap [i] = \emptyset$ ). The  $i$ -th *trace* of  $\pi$  is defined by

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \dots - B_k(\leq i).$$

We can represent  $T_i(\pi)$  by a graph  $D_i(\pi)$  on the vertex set  $[i]$ . Define  $D_i(\pi)$  as the subgraph of the graph of  $\pi$  induced by the vertex set  $[i]$ , with the additional condition that for any edge  $(x, y)$  of  $\pi$  such that  $x \leq i < y$ , we attach a "half-edge" to the vertex  $x$ , called *vacant vertex*. Denote by  $l_i(\pi)$  the number of vacant vertices in  $D_{i-1}(\pi)$ , with  $D_0 = \emptyset$ . Moreover, if  $i$  is a closer or a transient, there is an edge  $(j, i)$  with  $j < i$ , we denote by  $\gamma_i(\pi)$  the rank of the vertex  $j$  among the vacant vertices of  $D_{i-1}(\pi)$ , the vacant vertices being arranged from left to right in the order of their creation, namely, in increasing order.

For instance, if  $\pi$  is the partition given in Figure 1, then  $T_6(\pi) = \{1, \dots\} - \{2, 3, \dots\} - \{4\} - \{5, 6, \dots\}$ , where each opened block has an ellipsis. The corresponding graphs  $D_5(\pi)$  and  $D_6(\pi)$  are presented in Figure 4. We have  $l_6(\pi) = 3$  and  $\gamma_6(\pi) = 3$ .

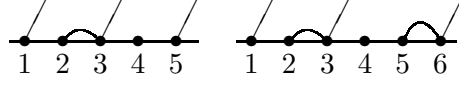


Figure 4: Graphs of  $D_5(\pi)$  and  $D_6(\pi)$

Now, we can describe our fundamental bijection  $\varphi$  using partition graphs. In the following, by "declare the vertex  $i$  vacant" we mean "attach a half-edge to the vertex  $i$ ". Let  $\pi \in \Pi_n$ , with type  $\lambda = (\mathcal{O}, \mathcal{C}, \mathcal{S}, \mathcal{T})$ . We obtain  $\varphi(\pi)$  by the following algorithm:

1. Set  $D'_0 = \emptyset$ .
2. For  $1 \leq i \leq n$ , the graph  $D'_i$  is obtained from  $D'_{i-1}$  by adding  $i$  as follows:
  - (i) if  $i \in \mathcal{O}$ , declare the vertex  $i$  vacant.
  - (ii) if  $i \in \mathcal{S}$ , add  $i$  as an isolated vertex.
  - (iii) if  $i \in \mathcal{C} \cup \mathcal{T}$ , join  $i$  to the  $\gamma_i(\pi)$ -th (from right to left) vacant vertex of  $D'_{i-1}$ . Moreover, if  $i \in \mathcal{T}$ , declare the vertex  $i$  vacant.
3. Set  $\varphi(\pi) := D'_n$

**Lemma 2.1** *The mapping  $\varphi : \Pi_n \rightarrow \Pi_n$  is well defined. Moreover, it is an involution which preserves the type.*

*Proof.* By induction on  $i$ , it is easy to see that  $D'_i$  ( $0 \leq i \leq n$ ) has the same vacant vertices as  $D_i(\pi)$ . So (iii) is valid and  $D'_n$  is a partition graph of  $[n]$ . The algorithm is well defined. By inspecting the algorithm, we see that  $\varphi(\pi)$  has the same type as  $\pi$ . Finally, the operation "reverse the order of the vacant vertices twice" preserves the original order. So  $\varphi$  is an involution. ■

**Remark 2.1** *The graph  $D'_i$  corresponds with the graph of  $i$ -th trace of  $\varphi(\pi)$ .*

For instance, if  $\pi$  is that in Figure 1, then  $\varphi(\pi) = \{1, 3, 10\} - \{2, 6, 9, 11\} - \{4\} - \{5, 7\} - \{8\}$ . Notice that  $\text{cr}(\varphi(\pi)) = \text{ne}(\pi) = 5$ ,  $\text{ne}(\varphi(\pi)) = \text{cr}(\pi) = 2$  and  $\text{al}(\varphi(\pi)) = \text{al}(\pi) = 8$ . An illustration of the step-by-step construction of  $\varphi(\pi)$  is given in Figure 5.

To complete the proof of Theorem 1.1 it remains to verify (1.1). In fact we shall prove a stronger result. For any closer or transient  $j$  of a partition  $\pi$ , let  $\text{cr}(\pi; j)$  (resp.  $\text{ne}(\pi; j)$  and  $\text{al}(\pi; j)$ ) be the number of crossings (resp. nestings and alignments) whose initial (resp. interior and initial) edge has  $j$  as the right-hand endpoint. Clearly

$$\text{cr}(\pi) = \sum \text{cr}(\pi; j), \quad \text{ne}(\pi) = \sum \text{ne}(\pi; j), \quad \text{al}(\pi) = \sum \text{al}(\pi; j),$$

where the summations are over  $j \in C(\pi) \cup T(\pi)$ .

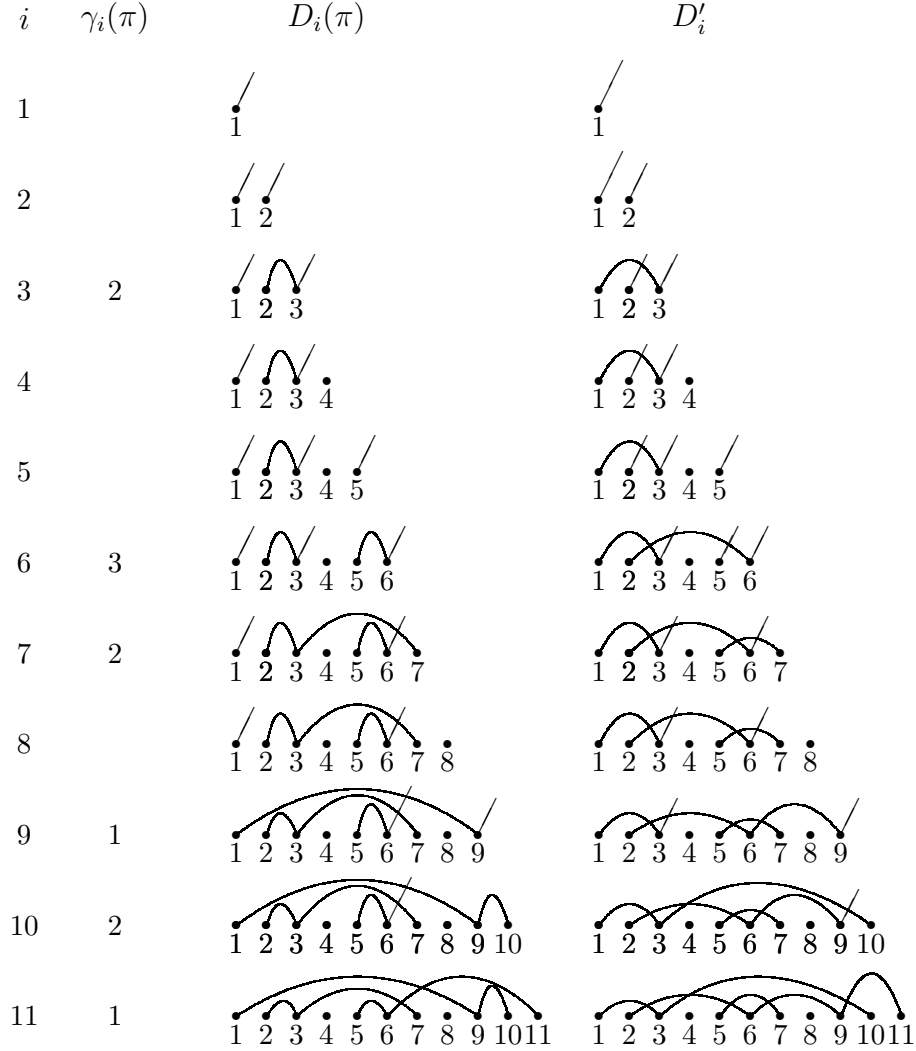


Figure 5: Construction of  $\varphi(\pi) = \{1, 3, 10\} - \{2, 6, 9, 11\} - \{4\} - \{5, 7\} - \{8\}$

**Lemma 2.2** *Let  $\pi$  be a partition of  $[n]$  and  $j$  a closer or transient of  $\pi$ . Then*

$$\text{al}(\varphi(\pi); j) = \text{al}(\pi; j), \quad \text{cr}(\varphi(\pi); j) = \text{ne}(\pi; j), \quad \text{ne}(\varphi(\pi); j) = \text{cr}(\pi; j).$$

*Proof.* For any partition  $\pi$ , the number of alignments with  $j$  as the right-hand endpoint, i.e.  $\text{al}(\pi; j)$ , is equal too the number of openers and transients which are  $\geq j$ . Now, as  $\varphi(\pi)$  has the same openers and transients as  $\pi$ , we get immediately  $\text{al}(\varphi(\pi); j) = \text{al}(\pi; j)$ .

Next, in the  $j$ -th ( $1 \leq j \leq n-1$ ) step of the construction of  $\varphi(\pi)$ , we add the vertex  $j$  to  $D'_{j-1}$  for obtaining  $D'_j$ . There are exactly  $l_j := l_j(\pi)$  vacant vertices in  $D'_{j-1}$  (resp.  $D_{j-1}$ ). These vertices are smaller than  $j$  and arranged from left to right in increasing order. Suppose that  $j$  is linked with the  $\gamma_j$ -th vacant vertex  $\bar{j}$  of  $D_{j-1}$  in  $D_j$  (resp.  $D'_{j-1}(\pi)$  in  $D'_j$ ). Recall that the rank of vacant vertices is counted from left to right in  $D_{j-1}$  and from right to left in  $D'_{j-1}$ .

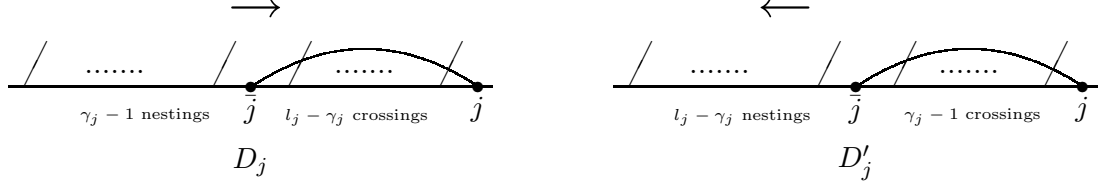


Figure 6: Counting of  $\text{cr}(\pi; j)$  and  $\text{cr}(\varphi(\pi); j)$

- Any vacant vertex  $\alpha$  on the left of the vertex  $z$  in  $D_j$  (resp.  $D'_j$ ) will be linked to a vertex  $\beta$  on the right of the vertex  $j$ ; thus  $(\alpha, \beta)$  will form a nesting with  $(\bar{j}, j)$  as an interior edge. Conversely, if  $(a, b)$  forms a nesting with interior edge  $(\bar{j}, j)$ , then  $a$  must be a vacant vertex on the left of the vertex  $\bar{j}$  in  $D_j$  (resp.  $D'_j$ ). We deduce that  $\text{ne}(\pi; j) = \gamma_j - 1$  and  $\text{ne}(\varphi(\pi); j) = l_j - \gamma_j$ .
- Any vacant vertex  $\alpha$  on the right of the vertex  $\bar{j}$  in  $D_j$  (resp.  $D'_j$ ) will be linked to a vertex  $\beta$  on the right of the vertex  $j$ ; thus  $(\alpha, \beta)$  will form a crossing with initial edge  $(\bar{j}, j)$ . Conversely, if  $(a, b)$  forms a crossing with initial edge  $(\bar{j}, j)$ , then the vertex  $a$  must be a vacant vertex on the right of the vertex  $\bar{j}$  in  $D_j$  (resp.  $D'_j$ ). We deduce that  $\text{cr}(\pi; j) = l_j - \gamma_j$  and  $\text{cr}(\varphi(\pi); j) = \gamma_j - 1$ .

The proof is completed by comparing the above counting results. ■

### 3 Factorization of $\varphi$ via Charlier diagrams

#### 3.1 Charlier diagrams

A *path* of length  $n$  is a finite sequence  $w = (s_0, s_1, \dots, s_n)$  of points  $s_i = (x_i, y_i)$  in the plan  $\mathbb{Z} \times \mathbb{Z}$ . A step  $(s_i, s_{i+1})$  of  $w$  is *East* (resp. *North-East* and *South-East*) if  $s_{i+1} = (x_i + 1, y_i)$  (resp.  $s_{i+1} = (x_i + 1, y_i + 1)$  and  $s_{i+1} = (x_i + 1, y_i - 1)$ ). The number  $y_i$  is the *height* of the step  $(s_i, s_{i+1})$ . The integer  $i + 1$  is the *index* of the step  $(s_i, s_{i+1})$ .

A *Motzkin path* is a path  $w = (s_0, s_1, \dots, s_n)$  such that:  $s_0 = (0, 0)$  and  $s_n = (n, 0)$ , each step is East or North-East or South-East and  $y_i \geq 0$  for each  $i$ . A *bicolored Motzkin* (or BM) path is a Motzkin path whose East steps are colored with *red* or *blue*. A *restricted bicolored Motzkin* (or RBM) path is a BM path whose blue East steps are of height  $> 0$ .

In the following, we shall write *BE*, *RE*, *NE* and *SE* as abbreviations of Blue East, Red East, North-East and South-East.

The type of  $w$  is the 4-tuple  $\lambda(w) = (\mathcal{O}(w), \mathcal{C}(w), \mathcal{S}(w), \mathcal{T}(w))$ , where  $\mathcal{O}(w)$  (resp.  $\mathcal{C}(w)$ ,  $\mathcal{S}(w)$ ,  $\mathcal{T}(w)$ ) is the set of indices of NE (resp. SE, RE, BE) steps of  $w$ . For instance, if  $w$  is the path in Figure 7, then

$$\lambda(w) = (\{1, 2, 5\}, \{7, 10, 11\}, \{4, 8\}, \{3, 6, 9\}).$$

Denote by  $M_b(n)$  (resp.  $M_{rb}(n)$ ) the set of BM (resp. RBM) paths of length  $n$ .

**Definition 3.1** A Charlier diagram of length  $n$  is a pair  $h = (w, \xi)$  where  $w = (s_0, \dots, s_n)$  is a RBM path and  $\xi = (\xi_1, \dots, \xi_n)$  is a sequence of integers such that  $\xi_i = 1$  if the  $i$ -th step is NE or RE, and  $1 \leq \xi_i \leq k$  if the  $i$ -th step is SE or BE of height  $k$ .

Let  $\Gamma_n$  be the set of Charlier diagrams of length  $n$ . A Charlier diagram is given in Figure 7.

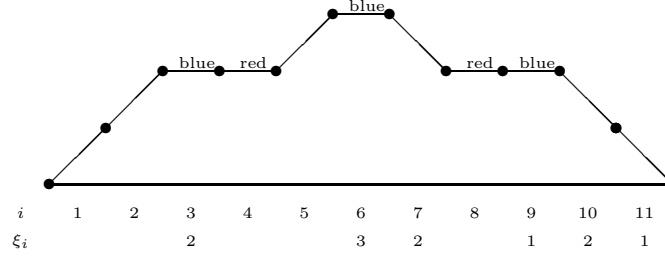


Figure 7: a Charlier diagram of length 11

There is a well-known bijection (see [6, 14]) from  $\Gamma_n$  to  $\Pi_n$ . For our purpose, we present two variants  $\varphi_l$  and  $\varphi_r$  of this bijection, which keep the track of crossings and nestings.

Let  $\pi$  be a partition. We denote respectively by  $\text{sg}(\pi)$ ,  $\text{bl}(\pi)$  and  $\text{tr}(\pi)$  the numbers of singletons, blocks whose cardinal is  $\geq 2$  and transients of the partition  $\pi$ .

A partition  $\pi$  of  $[n]$  is completely determined by its type  $\lambda = (\mathcal{O}, \mathcal{C}, \mathcal{S}, \mathcal{T})$  and the integers  $\gamma_i(\pi)$ ,  $i \in \mathcal{C} \cup \mathcal{T}$ . The description of the bijection  $\varphi_l$  is based on this fact. Given a Charlier diagram  $h = (w, \xi)$  of length  $n$ , we define the partition  $\pi = \varphi_l(h)$  as follows: the type of  $\pi$  is that of  $w$  and  $\gamma_j(\pi) := \xi_j$ , for  $j \in \mathcal{C} \cup \mathcal{T}$ .

In the definition of  $\varphi_l$ , we take  $\xi_j$  as the rank from left to right of the vacant vertex linked to  $j$  in the  $j$ -th step of the construction of  $\pi$ . If we take  $\xi_j$  as the rank from right to left of the vacant vertex linked to  $j$  in the  $j$ -th step of the construction of  $\pi$ , then we get the bijection  $\varphi_r$ . That is, we have  $\varphi_r := \varphi \circ \varphi_l$ . In other words, the following diagram is commutative.

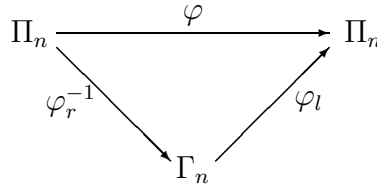


Figure 8: factorization of  $\varphi$

The following result is clear (cf. [6, 14]).

**Proposition 3.2** The mapping  $\varphi_l$  (resp.  $\varphi_r$ ) :  $\Gamma_n \rightarrow \Pi_n$  is a bijection. Moreover, if  $h = (w, \xi)$  is a Charlier diagram and  $\pi = \varphi_r(h)$  (or  $\pi = \varphi_l(h)$ ), then  $\text{sg}(\pi)$  (resp.  $\text{bl}(\pi)$ )



and  $\text{tr}(\pi)$ ) is equal to the number of red East (resp. North-East and blue East) steps of  $w$ .

For instance, if  $h = (w, \xi)$  is the Charlier diagram of Figure 7, the construction of  $\varphi_l(h)$  (resp.  $\varphi_r(h)$ ) correspond with the traces sequence  $D_i(\pi)$  (resp.  $D'_i$ ) in Figure 5. So,  $\varphi_l(h) = \{1, 9, 10\} - \{2, 3, 7\} - \{4\} - \{5, 6, 11\} - \{8\}$  and  $\varphi_r(h) = \{1, 3, 10\} - \{2, 6, 9, 11\} - \{4\} - \{5, 7\} - \{8\}$ .

**Proposition 3.3** *Let  $h = (w, \xi)$  be a Charlier diagram such that the  $j$ -th step of  $w$  is blue East or South-East of height  $k$ , then*

$$\begin{aligned} \text{cr}(\varphi_r(h); j) &= \text{ne}(\varphi_l(h); j) = \xi_j - 1 \\ \text{ne}(\varphi_r(h); j) &= \text{cr}(\varphi_l(h); j) = k - \xi_j \end{aligned}$$

*Proof.* This follows from the proof of Lemma 2.2 by replacing  $\varphi_l(h)$  by  $\pi$ ,  $\varphi_r(h)$  by  $\varphi(\pi)$ ,  $l_j$  by  $k$  and  $\gamma_j$  by  $\xi_j$ .  $\blacksquare$

A partition  $\pi$  is *noncrossing* (resp. *nonnesting*) if  $\text{cr}(\pi) = 0$  (resp.  $\text{ne}(\pi) = 0$ ). Let  $NC_n$  (resp.  $NN_n$ ) be the set of noncrossing (resp. nonnesting) partitions of  $[n]$ .

**Corollary 3.4** *Let  $\mathbf{1}$  denote the  $n$ -tuple  $(1, 1, \dots, 1)$ . Then*

- (i) *The mapping  $w \mapsto \varphi_r((w, \mathbf{1}))$  is a bijection from  $M_{rb}(n)$  to  $NC_n$ .*
- (ii) *The mapping  $w \mapsto \varphi_l((w, \mathbf{1}))$  is a bijection from  $M_{rb}(n)$  to  $NN_n$ .*

*Proof.* Let  $h = (w, \xi)$  a restricted diagram and suppose that the  $j$ -th step of  $w$  is blue East or South-East. Then, Proposition 3.3 implies that  $\text{cr}(\varphi_r(h); j) = \text{ne}(\varphi_l(h); j) = \xi_j - 1$ . Thus the partition  $\varphi_r(h)$  (resp.  $\varphi_l(h)$ ) is noncrossing (resp. nonnesting) if and only if  $\xi_i = 1$  for each  $i$ .  $\blacksquare$

**Remark 3.1** *Corollary 3.4 gives another proof of the well-known fact (see [11] and [12, p.226]) that the cardinals of  $NC_n$  and  $NN_n$  are equal to the  $n$ -th Catalan number  $C_n = \frac{1}{n+1} \binom{2n}{n}$ . Moreover, the mapping  $\varphi = \varphi_l \circ \varphi_r^{-1} : NC_n \rightarrow NN_n$  is a bijection.*

## 3.2 Continued fraction expansions

Consider the enumerating polynomial of  $\Pi_n$ :

$$B_n(p, q, u_1, u_2, v) = \sum_{\pi \in \Pi_n} p^{\text{cr}(\pi)} q^{\text{ne}(\pi)} u_1^{\text{sg}(\pi)} u_2^{\text{bl}(\pi)} v^{\text{tr}(\pi)},$$

which is a generalization of  $n$ -th Bell numbers. Let

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad [n]_q = \frac{1 - q^n}{1 - q}.$$

It follows from Proposition 3.3 that

$$B_n(p, q, u_1, u_2, v) = \sum_{(w, \xi) \in \Gamma_n} \left( \prod_{j \in \mathcal{O}(w)} u_2 \right) \left( \prod_{j \in \mathcal{S}(w)} u_1 \right) \left( \prod_{j \in \mathcal{C}(w)} p^{\xi_j - 1} q^{k_j - \xi_j} \right) \left( \prod_{j \in \mathcal{T}(w)} p^{\xi_j - 1} q^{k_j - \xi_j} v \right), \quad (3.1)$$

where  $k_j$  is the height of the  $j$ -th step of  $w$ .

For any BM path  $w$ , define the *weight* of a step of  $w$  at height  $k$  by  $u$  (resp.  $[k]_{p,q}$ ,  $d[k]_{p,q}(1 - \delta_{0k})$ ,  $v$ ) if it is NE (resp. SE, BE, RE) and the weight  $P(w)$  of  $w$  as the product of weights of its steps. We can rewrite the double sums in (3.1) as a single sum on bicolored Motzkin paths:

$$B_n(p, q, u_1, u_2, v) = \sum_{w \in M_b(n)} P(w).$$

Applying a well-known result of Flajolet [6, Propositions 7A and 7B], we derive immediately the continued fraction expansion from the above correspondence.

**Proposition 3.5** *The generating function  $\sum_{n \geq 0} B_n(p, q, u_1, u_2, v) z^n$  has the following continued fraction expansion:*

$$\cfrac{1}{1 - u_1 z - \cfrac{u_2 z^2}{1 - (u_1 + v)z - \cfrac{u_2 [2]_{p,q} z^2}{1 - (u_1 + [2]_{p,q} v)z - \cfrac{u_2 [3]_{p,q} z^2}{1 - (u_1 + [3]_{p,q} v)z - \cfrac{u_2 [4]_{p,q} z^2}{\dots}}}}}}.$$

Note that the  $q = v = 1$  case of Proposition 3.5 has been given by Biane [1]. Taking  $u_1 = u_2 = v = 1$ , we have:

**Corollary 3.6** *The generating function*

$$\sum_{n \geq 0} \left( \sum_{\pi \in \Pi_n} p^{\text{cr}(\pi)} q^{ne(\pi)} \right) z^n = \sum_{n \geq 0} \left( \sum_{\pi \in \Pi_n} q^{\text{cr}(\pi)} p^{ne(\pi)} \right) z^n$$

*has the following continued fraction expansion:*

$$\cfrac{1}{1 - z - \cfrac{z^2}{1 - ([1]_{p,q} + 1)z - \cfrac{[2]_{p,q} z^2}{1 - ([2]_{p,q} + 1)z - \cfrac{[3]_{p,q} z^2}{1 - ([3]_{p,q} + 1)z - \cfrac{[4]_{p,q} z^2}{\dots}}}}}}.$$

For any  $\pi \in \Pi_n$ , denote by  $\text{ed}(\pi)$  the number of edges of  $\pi$ . Clearly we have  $\text{ed}(\pi) = \text{bl}(\pi) + \text{tr}(\pi)$  and  $\text{cr}(\pi) + \text{ne}(\pi) + \text{al}(\pi) = \binom{\text{ed}(\pi)}{2}$ . Let

$$E_n(v, q) := \sum_{\pi \in \Pi_n} q^{\text{cr}(\pi) + \text{ne}(\pi)} v^{\text{ed}(\pi)}.$$

Setting  $p = q$ ,  $u_1 = 1$  and  $u_2 = v$  in Proposition 3.5, we get

**Corollary 3.7** *The generating function  $\sum_{n \geq 0} E_n(v, q) z^n$  has the following continued fraction expansion:*

$$\cfrac{1}{1 - z - \cfrac{vz^2}{1 - (1 + v)z - \cfrac{2qvz^2}{1 - (2qv + 1)z - \cfrac{3q^2vz^2}{1 - (3q^2v + 1)z - \cfrac{4q^3vz^2}{\dots}}}}}$$

Let  $E_n(v, q) = \sum_{k \geq 0} e_k(q) v^k$ . Then

$$F_n(q) := \sum_{\pi \in \Pi_n} q^{\text{al}(\pi)} = \sum_{k \geq 0} q^{\binom{k}{2}} e_k(1/q).$$

Finally consider the enumerating polynomials of crossings and nestings of  $\mathcal{M}_{2n}$ :

$$L_n(p, q) = \sum_{\alpha \in \mathcal{M}_{2n}} p^{\text{cr}(\alpha)} q^{\text{ne}(\alpha)} = \sum_{\alpha \in \mathcal{M}_{2n}} p^{\text{ne}(\alpha)} q^{\text{cr}(\alpha)}.$$

Setting  $u_2 = 1$ ,  $u_1 = v = 0$  in Proposition 3.5 and replacing  $z^2$  by  $z$  we get

**Proposition 3.8**

$$\sum_{n \geq 0} L_n(p, q) z^n = \cfrac{1}{1 - \cfrac{z}{1 - \cfrac{[2]_{p,q}z}{1 - \cfrac{[3]_{p,q}z}{1 - \cfrac{[4]_{p,q}z}{\dots}}}}}$$

Note that the  $p = 1$  case of Proposition 3.8 corresponds to a result of Touchard [13]. Since a matching of  $[2n]$  has exactly  $n$  edges, we get  $\text{cr}(\alpha) + \text{ne}(\alpha) + \text{al}(\alpha) = \binom{n}{2}$  for any  $\alpha \in \mathcal{M}_{2n}$ . Therefore

$$T_n(q) := \sum_{\alpha \in \mathcal{M}_{2n}} q^{\text{al}(\alpha)} = q^{\binom{n}{2}} L_n(1/q, 1/q).$$

The first terms of the above sequences are given as follows:

$$\begin{aligned} T_0(q) &= T_1(q) = 1 & L_0(p, q) &= L_1(p, q) = 1 \\ T_2(q) &= 2 + q & L_2(p, q) &= 1 + p + q \\ T_3(q) &= 6 + 4q + 4q^2 + q^3 & L_3(p, q) &= 1 + 2p + 2q + 2pq + p^2 + q^2 + 2p^2q + 2pq^2 + p^3 + q^3. \end{aligned}$$

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